

TRUNCATED TOEPLITZ OPERATORS OF FINITE RANK

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ABSTRACT. We give a complete description of the finite-rank truncated Toeplitz operators.

1. INTRODUCTION

Truncated Toeplitz operators are compressions of multiplication operators on L^2 to model subspaces $K_\theta = H^2 \ominus \theta H^2$ of the Hardy class H^2 , where θ is an inner function. In [1] D.Sarason gave a characterization of all truncated Toeplitz operators of rank one. Moreover, he constructed a class of finite-rank truncated Toeplitz operators and asked whether this class exhausts all truncated Toeplitz operators having finite rank. Our main result, Theorem 1, answers this question in the affirmative.

1.1. Definitions. As usual, we identify the Hardy class H^2 in the unit disk \mathbb{D} with the subspace of the space L^2 on the unit circle \mathbb{T} via non-tangential boundary values. A function $\theta \in H^2$ is called inner if $|\theta(z)| = 1$ almost everywhere with respect to the Lebesgue measure on \mathbb{T} . Denote by P_θ the orthogonal projection from L^2 onto K_θ . The truncated Toeplitz operator A_φ with symbol $\varphi \in L^2$ is the mapping

$$(1) \quad A_\varphi : f \mapsto P_\theta(\varphi f), \quad f \in K_\theta \cap L^\infty.$$

We deal only with bounded truncated Toeplitz operators. For $\lambda \in \mathbb{D}$ define

$$(2) \quad k_\lambda = \frac{1 - \overline{\theta(\lambda)}\theta}{1 - \overline{\lambda}z}, \quad \tilde{k}_\lambda = \frac{\theta - \theta(\lambda)}{z - \lambda}.$$

The function k_λ is the reproducing kernel of the space K_θ at the point λ ; \tilde{k}_λ is the conjugate kernel at λ (see Section 2 for details). For an integer $n \geq 0$, denote by $\Omega_n = \Omega(\theta, n)$ the set of all points $\lambda \in \mathbb{T}$ such that every function from K_θ and its derivatives up to order n have non-tangential limits at λ . A description of Ω_n is given by P.R.Ahern and D.N. Clark in [2], we discuss it in Section 2. In particular, for $\lambda \in \mathbb{T}$, we have $\lambda \in \Omega_0$ if and only if $k_\lambda, \tilde{k}_\lambda \in K_\theta$.

The only compact Toeplitz operator on the Hardy space H^2 is the zero operator [3]. The situation is different for Toeplitz operators on K_θ . It was proved in [1]

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that the general rank-one truncated Toeplitz operator on K_θ is a scalar multiple of

$$(3) \quad k_\lambda \otimes \tilde{k}_\lambda \quad \text{or} \quad \tilde{k}_\lambda \otimes k_\lambda,$$

for some $\lambda \in \mathbb{D} \cup \Omega_0$, where we use the standard notation for rank-one operators in the Hilbert space: $x \otimes y : h \mapsto (h, y)x$. Some finite-rank truncated Toeplitz operators can be obtained from the rank-one operators (3) by differentiation. Consider the analytic mapping $\Phi : \lambda \mapsto \tilde{k}_\lambda \otimes k_\lambda$ from the unit disk \mathbb{D} into the space of all bounded operators on K_θ . Take an integer $n \geq 1$. Denote by $D^n[\tilde{k}_\lambda \otimes k_\lambda]$ the value $\Phi^{(n)}(\lambda)$ of the n -th derivative of Φ at the point $\lambda \in \mathbb{D}$. For $\lambda \in \Omega_n$ let $D^n[\tilde{k}_\lambda \otimes k_\lambda]$ denote the n -th angular derivative of Φ at λ . Existence of this derivative follows from results of P.R.Ahern and D.N.Clark and from representation (11) of $D^n[\tilde{k}_\lambda \otimes k_\lambda]$ in terms of derivatives of reproducing kernels; see Section 2.2 for details. Since the set of all truncated Toeplitz operators is linear space closed in the weak operator topology [1], the derivative $D^n[\tilde{k}_\lambda \otimes k_\lambda]$ determines truncated Toeplitz operator of rank n for every $\lambda \in \mathbb{D} \cup \Omega_n$. Parallel arguments work for the adjoint operator $\bar{D}^n[k_\lambda \otimes \tilde{k}_\lambda]$: it is the derivative of $k_\lambda \otimes \tilde{k}_\lambda$ of order n with respect to $\bar{\lambda}$. Operators of the form

$$(4) \quad \bar{D}^n[k_\lambda \otimes \tilde{k}_\lambda], \quad D^n[\tilde{k}_\lambda \otimes k_\lambda], \quad n \geq 0, \quad \lambda \in \mathbb{D} \cup \Omega_n,$$

were originally constructed in [1] as examples of finite-rank truncated Toeplitz operators that are not linear combinations of the rank-one operators (3).

1.2. The main result.

Theorem 1. *The general finite-rank truncated Toeplitz operator on K_θ is a finite linear combination of the operators in (4).*

The key step in proving of Theorem 1 is the identification of range of a general finite-rank truncated Toeplitz operator. Set

$$F(\lambda, n) = \begin{cases} \text{Ran } \bar{D}^n[k_\lambda \otimes \tilde{k}_\lambda], & \text{if } \lambda \in \mathbb{D} \cup \Omega_n; \\ \text{Ran } D^n[\tilde{k}_{\lambda^*} \otimes k_{\lambda^*}], & \text{if } \lambda \in \mathbb{D}_e, \end{cases}$$

where $\mathbb{D}_e = \{z : |z| > 1\} \cup \{\infty\}$, $\lambda^* = 1/\bar{\lambda}$ and $\infty^* = 0$. It will be shown in Section 2.3 that a subspace $F \subset K_\theta$ is range of an operator in (4) if and only if we have $F = F(\lambda, n)$ for some $n \geq 0$ and $\lambda \in \mathbb{D} \cup \Omega_n \cup \mathbb{D}_e$. Theorem 1 will be proved as soon as we establish the following three results.

Lemma 1.1. *Suppose A is a finite-rank truncated Toeplitz operator on K_θ . Then there exists a finite collection of points $\lambda_k \in \mathbb{D} \cup \Omega_{n_k} \cup \mathbb{D}_e$ such that*

$$(5) \quad \text{Ran } A = F(\lambda_1, n_1) \dot{+} \dots \dot{+} F(\lambda_s, n_s).$$

Lemma 1.2. *Suppose A is a truncated Toeplitz operator on K_θ with range of the form (5). Then A is a sum of s truncated Toeplitz operators A_k such that $\text{Ran } A_k = F(\lambda_k, n_k)$.*

Lemma 1.3. *Suppose A is a truncated Toeplitz operator on K_θ with the range $\text{Ran } A = F(\lambda, n)$, where $\lambda \in \mathbb{D} \cup \Omega_n \cup \mathbb{D}_e$. Then A is a finite linear combination of the operators in (4).*

In Section 2 we collect some standard results about the space K_θ . Section 3 concerns a description of range of a general bounded truncated Toeplitz operator. Lemmas 1.1, 1.2, 1.3 will be proved in Section 4.

1.3. A duality approach. The rank-one truncated Toeplitz operators can be regarded as a point evaluation in a special Banach space of analytic functions. Given an inner function θ , define the space X_a by

$$X_a = \left\{ \sum_0^\infty x_k y_k : x_k, y_k \in K_\theta \text{ and } \sum_0^\infty \|x_k\| \cdot \|y_k\| < \infty \right\}.$$

This space was introduced in [4], where the fact that X_a is the predual of the space of all bounded truncated Toeplitz operators established. The pairing is given by

$$(6) \quad \langle A, \sum x_k y_k \rangle = \sum (Ax_k, y_k).$$

It was shown in [4], that $\langle \tilde{k}_\lambda \otimes k_\lambda, xy \rangle = (x\tilde{y})(\lambda)$, where $\tilde{y} = \bar{z}\theta\bar{y} \in K_\theta$. This yields the nice formula

$$\langle D^n[\tilde{k}_\lambda \otimes k_\lambda], \sum x_k y_k \rangle = \left(\sum x_k \tilde{y}_k \right)^{(n)}(\lambda).$$

A similar relation holds for the operators $\bar{D}^n[k_\lambda \otimes \tilde{k}_\lambda]$. In the sense of the pairing (6) the operator $k_\lambda \otimes \tilde{k}_\lambda$ is the point evaluation at $\lambda^* = 1/\bar{\lambda}$ of functions from X_a up to the scalar $(\lambda^*/\theta(\lambda^*))^2$, and $\bar{D}^n[k_\lambda \otimes \tilde{k}_\lambda]$ is its derivative with respect to $\bar{\lambda}$. Since $|\lambda^*| \geq 1$, the point evaluation should be understood in terms of pseudocontinuations of functions from K_θ to the exterior of the unit disk (see Lecture II in [5]).

1.4. Notations.

- \mathbb{Z}_+ is the set of nonnegative integers;
- \mathcal{T}_θ is the linear space of all bounded truncated Toeplitz operators on K_θ ;
- $\text{Ker } A$ is kernel of a bounded operator A ;
- $\text{Ran } A$ is range of a bounded operator A ;
- $\overline{\text{Ran } A}$ is the closure of $\text{Ran } A$;
- $\langle f_1 \dots f_n \rangle = \text{span}\{f_j, j = 1 \dots n\}$;
- H^\perp is the orthogonal complement to a subspace H ;
- $H_1 + H_2$ is the linear span of the union $H_1 \cup H_2$;
- $H_1 \dot{+} H_2$ is the direct sum of two subspaces H_1, H_2 .

2. PRELIMINARIES

This section contains the preliminary information concerning spaces K_θ and truncated Toeplitz operators: reproducing kernels, Clark unitary perturbations, Sarason's characterization of truncated Toeplitz operators. A more detailed discussion is available in [1], [5], [6].

2.1. The conjugation. The space K_θ is closed under the conjugation

$$(7) \quad C : x \mapsto \bar{z}\theta\bar{x}.$$

Truncated Toeplitz operators are complex symmetric with respect to C , which means $CA = A^*C$, see [1]. Hence the ranges of A and A^* are mutually conjugate for every operator $A \in \mathcal{T}_\theta$: $\text{Ran } A^* = C \text{Ran } A$. For more information on this property of truncated Toeplitz operators see [7], [8].

2.2. Reproducing kernels and their derivatives. For $\lambda \in \mathbb{D}$, set

$$(8) \quad k_\lambda = \frac{1 - \overline{\theta(\lambda)}\theta}{1 - \bar{\lambda}z}, \quad \tilde{k}_\lambda = \frac{\theta - \theta(\lambda)}{z - \lambda}.$$

Note that $\tilde{k}_\lambda = Ck_\lambda$, where the conjugation C is defined by (7). The function k_λ (respectively, \tilde{k}_λ) is the reproducing kernel (respectively, conjugate reproducing kernel) of the space K_θ at the point λ :

$$(9) \quad (f, k_\lambda) = f(\lambda), \quad (f, \tilde{k}_\lambda) = \overline{(Cf)(\lambda)}.$$

Differentiating (9), we obtain

$$(10) \quad (f, \bar{\partial}^n k_\lambda) = f^{(n)}(\lambda), \quad (f, \partial^n \tilde{k}_\lambda) = \overline{(Cf)^{(n)}(\lambda)}.$$

In what follows the symbols $\bar{\partial}^n k_\lambda$ and $\partial^n \tilde{k}_\lambda$ denote the n -th derivatives of $k_\lambda, \tilde{k}_\lambda$ with respect to $\bar{\lambda}, \lambda$, respectively. For example, in the case $n = 1$ we have

$$\bar{\partial} k_\lambda = \lim_{\mu \rightarrow \lambda} \frac{k_\lambda - k_\mu}{\bar{\lambda} - \bar{\mu}}, \quad \partial \tilde{k}_\lambda = \lim_{\mu \rightarrow \lambda} \frac{\tilde{k}_\lambda - \tilde{k}_\mu}{\lambda - \mu}.$$

Let $\theta = B_\Lambda S_\Lambda$ be an inner function with the set of zeroes $\Lambda = (a_k)_{k=1}^N$, repeated according to multiplicity, and the singular part S_ν that corresponds to a singular measure ν on the unit circle \mathbb{T} . Take a point $\lambda \in \mathbb{T}$ and an integer $n \in \mathbb{Z}_+$. The following result is in [2], see also [9].

Theorem 2 (P.R.Ahern and D.N.Clark). *The following are equivalent:*

- (a) *functions $f, f', \dots, f^{(n)}$ have non-tangential limits at λ for every $f \in K_\theta$;*
- (b) *the non-tangential limits $\bar{\partial}^j k_\lambda = \lim_{\mu \rightarrow \lambda} \bar{\partial}^j k_\mu$ and $\partial^j \tilde{k}_\lambda = \lim_{\mu \rightarrow \lambda} \partial^j \tilde{k}_\mu$ exist in norm of K_θ for every $j = 0 \dots n$;*
- (c) *$\sum_{k=1}^N \frac{1 - |a_k|^2}{|1 - \bar{\lambda} \bar{a}_k|^{2n+2}} < \infty$ and $\int_{\mathbb{T}} \frac{d\nu(\xi)}{|1 - \bar{\lambda} \xi|^{2n+2}} < \infty$.*

Denote by $\Omega_n = \Omega(\theta, n)$ the set of all points $\lambda \in \mathbb{T}$ that satisfy the conditions of Theorem 2. For $\lambda \in \mathbb{D}$ we have

$$(11) \quad \begin{aligned} A_\varphi &= \bar{D}^n [k_\lambda \otimes \tilde{k}_\lambda] = \sum_{k=0}^n C_n^k \left(\bar{\partial}^k k_\lambda \otimes \partial^{n-k} \tilde{k}_\lambda \right), & \varphi(z) &= \frac{n! \cdot \overline{\theta(z)}}{(\bar{z} - \bar{\lambda})^{n+1}}; \\ A_\varphi &= D^n [\tilde{k}_\lambda \otimes k_\lambda] = \sum_{k=0}^n C_n^k \left(\partial^k \tilde{k}_\lambda \otimes \bar{\partial}^{n-k} k_\lambda \right), & \varphi(z) &= \frac{n! \cdot \theta(z)}{(z - \lambda)^{n+1}}. \end{aligned}$$

by using the Leibniz formula for derivatives of a bilinear expression. It follows from Theorem 2 that formulas (8)-(11) hold in the sense of non-tangential boundary values for every $\lambda \in \Omega_n$. Thus, the operators in (4) are exactly the operators in (11) for $n \in \mathbb{Z}_+$ and $\lambda \in \mathbb{D} \cup \Omega_n$.

2.3. The restricted shift. Consider the operator $S_\theta : f \mapsto P_\theta(zf)$. We have

$$(12) \quad \begin{aligned} S_\theta k_\lambda &= (k_\lambda - k_0)/\bar{\lambda}; & S_\theta \tilde{k}_\lambda &= \lambda \tilde{k}_\lambda - \theta(\lambda)k_0; \\ S_\theta k_0 &= \bar{\partial} k_0; & S_\theta \tilde{k}_0 &= -\theta(0)k_0, \end{aligned}$$

where $\lambda \neq 0$ is a point from $\mathbb{D} \cup \Omega_0$. In particular, if $\theta(0) = 0$ then $\tilde{k}_0 \in \text{Ker } S_\theta$. Actually, we have $\text{Ker } S_\theta = \langle \tilde{k}_0 \rangle$ in that case.

Proposition 2.1. *We have $S_\theta^n k_0 = \frac{1}{n!} \bar{\partial}^n k_0$ and $S_\theta^n \partial \tilde{k}_0 = n \partial^{n-1} \tilde{k}_0 + \theta^{(n)}(0)k_0$ for every integer $n \geq 1$.*

Proof. Take a function $f \in K_\theta$ and consider $(f, S_\theta^n k_0) = ((S_\theta^*)^n f, k_0)$. The space K_θ is invariant under the backward shift operator $S^* : f \mapsto (f - f(0))/z$, see [5]. Hence $S^*[K_\theta] = S_\theta^*$ and $((S_\theta^*)^n f, k_0) = \frac{1}{n!} f^{(n)}(0)$. It follows from (10) that $f^{(n)}(0) = (f, \bar{\partial}^n k_0)$, and therefore $(f, S_\theta^n k_0) = \frac{1}{n!} (f, \bar{\partial}^n k_0)$. Since this equality holds for every function $f \in K_\theta$, we obtain $S_\theta^n k_0 = \frac{1}{n!} \bar{\partial}^n k_0$. Differentiating the identity $S_\theta \tilde{k}_\lambda = \lambda \tilde{k}_\lambda - \theta(\lambda) k_0$ with respect to λ at the point $\lambda = 0$, we get the formula $S_\theta^n \partial \tilde{k}_0 = n \partial^{n-1} \tilde{k}_0 + \theta^{(n)}(0) k_0$. \square

Consider the subspaces $F(\lambda, n)$ defined in Section 1.2. It follows from formula (11) that

$$(13) \quad F(\lambda, n) = \begin{cases} \text{span}\{\bar{\partial}^j k_\lambda, j = 0 \dots n\}, & \text{if } \lambda \in \mathbb{D} \cup \Omega_n; \\ \text{span}\{\partial^j \tilde{k}_{\lambda^*}, j = 0 \dots n\}, & \text{if } \lambda \in \mathbb{D}_e. \end{cases}$$

For each point $\lambda \in \Omega_0$ we have $\lambda^* = \lambda$ and $\tilde{k}_\lambda = \bar{\lambda} \theta(\lambda) k_\lambda$. Therefore,

$$\text{span}\{\bar{\partial}^j k_\lambda, j = 0 \dots n\} = \text{span}\{\tilde{k}_{\lambda^*}, j = 0 \dots n\}$$

for all $\lambda \in \Omega_n$ and $n \in \mathbb{Z}_+$. We now see from (11) that range of the operator $\bar{D}^n[k_\lambda \otimes \tilde{k}_\lambda]$ coincides with range of the operator $D^n[\tilde{k}_\lambda \otimes k_\lambda]$ if $\lambda \in \Omega_n$. Therefore, a subspace $F \subset K_\theta$ is range of an operator in (4) if and only if we have $F = F(\lambda, n)$ for some $n \geq 0$ and $\lambda \in \mathbb{D} \cup \Omega_n \cup \mathbb{D}_e$, as claimed in Section 1.2.

Proposition 2.2. *We have*

$$(14) \quad \begin{aligned} S_\theta F(\lambda, n) &\subset F(\lambda, n) \dot{+} \langle k_0 \rangle, \text{ if } \lambda \neq 0; \\ S_\theta F(0, n) &\subset F(0, n) \dot{+} \langle \bar{\partial}^{n+1} k_0 \rangle. \end{aligned}$$

Proof. The first formula in (14) can be obtained from (12) by differentiation. The second one follows from Proposition 2.1. \square

2.4. The Frostman shift. Let θ be an inner function. The Frostman shift of θ corresponding to the point $\theta(0)$ is the inner function $\Theta = \frac{\theta - \theta(0)}{1 - \overline{\theta(0)}\theta}$. We have $\Theta(0) = 0$. Define the unitary operator $J : K_\theta \rightarrow K_\Theta$ by

$$J : f \mapsto \frac{\sqrt{1 - |\theta(0)|^2}}{1 - \overline{\theta(0)}\theta} f.$$

Proposition 2.3. *For $\lambda \in \mathbb{D}$ and $n \in \mathbb{Z}_+$ we have*

$$(15) \quad J \text{span}\{\bar{\partial}^j k_\lambda, j = 0 \dots n\} = \text{span}\{\bar{\partial}^j k_\lambda^\Theta, j = 0 \dots n\},$$

where k_λ^Θ is the reproducing kernel of the space K_Θ at the point λ .

Proof. The formula $Jk_\lambda = (1 - \theta(0)\overline{\theta(\lambda)})/(\sqrt{1 - |\theta(0)|^2})k_\lambda^\Theta$ follows from the definition of J and implies (15) by differentiation; see details in Section 13 of [1]. \square

The following fact is a particular case of Theorem 13.2 from [1].

Proposition 2.4. *A bounded operator A is a truncated Toeplitz operator on K_θ if and only if the operator JAJ^{-1} is a truncated Toeplitz operator on K_Θ .*

2.5. Clark's unitary perturbations. In [9] D.N.Clark described one-dimensional unitary perturbations of S_θ . Given a number $\alpha \in \mathbb{T}$, define

$$(16) \quad U_\alpha = S_\theta + c_\alpha k_0 \otimes \tilde{k}_0, \quad c_\alpha = \frac{\alpha + \theta(0)}{1 - |\theta(0)|^2},$$

where k_0, \tilde{k}_0 are the reproducing kernels (8) at the origin. The operators U_α are unitary and cyclic; every one-dimensional unitary perturbation of S_θ is U_α for an appropriate number $\alpha \in \mathbb{T}$, see [9]. It is shown in [9] that the spectral measure σ_α of the unitary operator U_α can be chosen so that

$$(17) \quad U_\alpha = V_\alpha^{-1} M_z V_\alpha,$$

where $V_\alpha : K_\theta \rightarrow L^2(\sigma_\alpha)$ is the unitary operator that sends functions from a dense subset of K_θ to their boundary values on \mathbb{T} ; M_z is the operator of multiplication by z on $L^2(\sigma_\alpha)$. A.G.Poltoratski [10] established the existence of non-tangential boundary values σ_α -almost everywhere for all functions from K_θ . Thus, the operator V_α is the well-defined unitary embedding $K_\theta \rightarrow L^2(\sigma_\alpha)$. For every function $f \in K_\theta$ we have

$$(18) \quad f(z) = \int_{\mathbb{T}} (V_\alpha f)(\xi) \frac{1 - \bar{\alpha}\theta(z)}{1 - \bar{\xi}z} d\sigma_\alpha(\xi), \quad z \in \mathbb{D}.$$

The following fact is due to D.N.Clark [9], see also [6].

Proposition 2.5 (D.N.Clark). *We have $\sigma_\alpha\{\xi \in \mathbb{T} : \theta(\xi) = \alpha\} = \sigma_\alpha(\mathbb{T})$ for each $\alpha \in \mathbb{T}$. The measure σ_α has an atom at a point $\lambda \in \mathbb{T}$ if and only if $\lambda \in \Omega_0$ and $\theta(\lambda) = \alpha$. In that case we have $V_\alpha^{-1}\mathbb{I}_{\{\lambda\}} = \sigma_\alpha(\{\lambda\}) \cdot k_\lambda$, where $\mathbb{I}_{\{\lambda\}}$ denotes the indicator of the singleton $\{\lambda\}$.*

Proposition 2.6. *If $(z - \lambda)^{-n-1} \in L^2(\sigma_\alpha)$ for some $\lambda \in \mathbb{T}$ and $n \in \mathbb{Z}_+$, then $\lambda \in \Omega_n$ and $\theta(\lambda) \neq \alpha$.*

Proof. At first, let $\theta(0) = 0$. In this case constants lie in the space K_θ . Formula (18) with $f \equiv 1$ gives us

$$(19) \quad \frac{1}{1 - \bar{\alpha}\theta(z)} = \int_{\mathbb{T}} \frac{d\sigma_\alpha(\xi)}{1 - \bar{\xi}z}, \quad z \in \mathbb{D}.$$

We have

$$(20) \quad \frac{1}{|1 - \bar{\xi}z|} \leq \frac{c}{|1 - \bar{\xi}\lambda|}, \quad \xi \in \mathbb{T},$$

for some constant c , as z tends non-tangentially to λ . Since $(1 - \bar{\xi}\lambda)^{-1} \in L^2(\sigma_\alpha)$, we see from (19) and (20) that the function θ has non-tangential limit at λ and $\theta(\lambda) \neq \alpha$. Similarly, differentiating (19) with respect to z , one can prove that $\theta', \theta'', \dots, \theta^{(n)}$ also have non-tangential limits at λ . In the case $\theta(0) \neq 0$ this fact follows from the consideration of the Frostman shift of the function θ .

Take $f \in K_\theta$ and consider $f^{(j)}$, where $0 \leq j \leq n$ is integer. It follows from (18) that

$$(21) \quad f^{(j)}(z) = \int_{\mathbb{T}} (V_\alpha f)(\xi) \left(\frac{1 - \bar{\alpha}\theta(z)}{1 - \bar{\xi}z} \right)^{(j)} d\sigma_\alpha(\xi), \quad z \in \mathbb{D}.$$

Since $(1 - \bar{\lambda}z)^{-j-1} \in L^2(\sigma_\alpha)$, we see from (20) and (21) that $f^{(j)}$ has the non-tangential limit at the point λ for every $j = 0 \dots n$. Thus, we have $\lambda \in \Omega_n$. \square

We now describe boundary values of functions from subspaces $F(\lambda, n)$ in (13).

Proposition 2.7. *Let $\alpha \in \mathbb{T}$, $n \in \mathbb{Z}_+$ and let $\lambda \in \mathbb{D} \cup \Omega_n \cup \mathbb{D}_e$, where in the case $|\lambda| = 1$ we assume that $\theta(\lambda) \neq \alpha$. We have*

$$(22) \quad V_\alpha F(\lambda, n) = \text{span}\{(z - \lambda^*)^{-j}, j = 1 \dots n + 1\}, \quad \lambda \neq 0, \lambda^* = 1/\bar{\lambda};$$

$$(23) \quad V_\alpha F(0, n) = \text{span}\{z^j, j = 0 \dots n\}.$$

Proof. In the case $|\lambda| \neq 1$ formula (22) follows from the definition of V_α . Formula (23) is a consequence of Proposition 2.1. Take a point $\lambda = \lambda^*$ from Ω_0 such that $\theta(\lambda) \neq \alpha$. By Proposition 2.5 we have $\sigma_\alpha(\{\lambda\}) = 0$ and

$$V_\alpha k_\lambda = \frac{1 - \overline{\theta(\lambda)}\alpha}{1 - \bar{\lambda}z}$$

σ_α -almost everywhere. Since $V_\alpha k_\lambda \in L^2(\sigma_\alpha)$, we have $(z - \lambda)^{-1} \in L^2(\sigma_\alpha)$ and therefore (22) holds in the case $n = 0$. For $n = 1$ and $\lambda \in \Omega_1$, consider

$$V_\alpha \bar{\partial} k_\lambda = \frac{-\overline{\theta'(\lambda)}\alpha}{1 - \bar{\lambda}z} + \frac{z(1 - \overline{\theta(\lambda)}\alpha)}{(1 - \bar{\lambda}z)^2}.$$

Since $V \bar{\partial} k_\lambda \in L^2(\sigma_\alpha)$ and $(z - \lambda)^{-1} \in L^2(\sigma_\alpha)$, we have $(z - \lambda)^{-2} \in L^2(\sigma_\alpha)$. Hence formula (22) holds in the case $n = 1$. Arguing as above, we prove (22) for all $\lambda \in \Omega_n$ and $n \in \mathbb{Z}_+$. \square

Proposition 2.8. *Suppose that an inner function θ is not a finite Blaschke product. Then every finite collection of the functions $\bar{\partial}^{s_k} k_{\lambda_k}$, $\partial^{t_k} \bar{k}_{\lambda_k}$, where $\lambda_k \in \mathbb{D} \cup \Omega_{s_k}$ and $\lambda_k \in \mathbb{D} \cup \Omega_{t_k}$, is linearly independent in K_θ .*

Proof. Since θ is not a finite Blaschke product, the space K_θ has infinite dimension, see Lecture II in [5]. Hence the space $L^2(\sigma_\alpha)$, $|\alpha| = 1$, has infinite dimension as well. The result now follows from Proposition 2.7. \square

2.6. A characterization. In what follows we will often use the following characterization of truncated Toeplitz operators.

Theorem 3 (D.Sarason, [1]). *A bounded operator A on K_θ is a truncated Toeplitz operator if and only if there exist functions ψ, χ in K_θ such that*

$$(24) \quad A - S_\theta A S_\theta^* = \psi \otimes k_0 + k_0 \otimes \chi,$$

in which case $A = A_{\psi + \bar{\chi}}$.

Remark 2.9. *Suppose $\theta(0) = 0$; then $\psi = A k_0 - \overline{\chi(0)} k_0$.*

Proof. Apply both sides of (24) to the vector $k_0 \equiv 1$ and use the relation $S_\theta^* 1 = 0$. \square

3. THE RANGE OF A BOUNDED TRUNCATED TOEPLITZ OPERATOR

In this section we prove the following result.

Proposition 3.1. *Let $A \in \mathcal{T}_\theta$, and assume that $\overline{\text{Ran } A} \neq K_\theta$. Then*

$$(25) \quad S_\theta \overline{\text{Ran } A} \subset \overline{\text{Ran } A} + \langle \bar{\partial}^n k_0 \rangle,$$

where $n \in \mathbb{Z}_+$ is the maximal integer such that $\bar{\partial}^j k_0 \in \overline{\text{Ran } A}$ for every $0 \leq j < n$.

For the proof we need two lemmas.

Lemma 3.2. *Let $\theta(0) = 0$, $A \in \mathcal{T}_\theta$. Then*

- (1) $S_\theta A \left[\langle \tilde{k}_0 \rangle^\perp \right] \subset \overline{\text{Ran } A} + \langle k_0 \rangle$;
- (2) $S_\theta \overline{\text{Ran } A} \subset \overline{\text{Ran } A} + \langle k_0, g \rangle$ for some $g \in K_\theta$.

Proof. Indeed, by Theorem 3 and Remark 2.9 we have $S_\theta A S_\theta^* h \in \overline{\text{Ran } A} + \langle k_0 \rangle$ for every vector $S_\theta^* h$ from $\overline{\text{Ran } S_\theta^*} = \langle \tilde{k}_0 \rangle^\perp$. This proves assertion (1). To prove (2), one can take $g = S_\theta A \tilde{k}_0$. \square

Lemma 3.3. *Let $\theta(0) = 0$, $A \in \mathcal{T}_\theta$. Assume that $k_0 \notin \overline{\text{Ran } A}$. Then*

$$S_\theta \overline{\text{Ran } A} \subset \overline{\text{Ran } A} + \langle k_0 \rangle.$$

Proof. Denote by P_1, P_2 the orthogonal projections on K_θ with the ranges $\overline{\text{Ran } A^*}$ and $\langle \tilde{k}_0 \rangle$, respectively. Note that $A = AP_1$. For every $h \in K_\theta$ we have

$$S_\theta Ah = S_\theta AP_1 h = S_\theta AP_2 P_1 h + f_1,$$

where $f_1 = S_\theta AP_2^\perp P_1 h$. It follows from assertion (1) of Lemma 3.2 that $f_1 \in \overline{\text{Ran } A} + \langle k_0 \rangle$. Proceeding inductively, we obtain the relation

$$(26) \quad S_\theta Ah = S_\theta A(P_2 P_1)^n h + f_n$$

for some $f_n \in \overline{\text{Ran } A} + \langle k_0 \rangle$. Since $\text{Ran } A^* = C \text{Ran } A$ (see Section 2.1) and $k_0 \notin \overline{\text{Ran } A}$, we have $\tilde{k}_0 \notin \overline{\text{Ran } A^*}$, and therefore $\|P_2 P_1\| < 1$. Passing to the limit in (26), we see that $S_\theta Ah \in \overline{\text{Ran } A} + \langle k_0 \rangle$ and the result follows. \square

Proof of Proposition 3.1. At first, let $\theta(0) = 0$. In the case $k_0 \notin \overline{\text{Ran } A}$, Lemma 3.3 gives us formula (25) with $n = 0$. Now suppose that $k_0 \in \overline{\text{Ran } A}$. Since the family $\{\partial^j k_0\}_0^\infty$ is complete in K_θ (see formula 10), one can choose the maximal integer $n \geq 1$ such that $\partial^j k_0 \in \overline{\text{Ran } A}$ for every $0 \leq j < n$ and $\partial^n k_0 \notin \overline{\text{Ran } A}$. It follows from Proposition 2.1 and Lemma 3.2 that

$$(27) \quad \bar{\partial}^n k_0 = S_\theta \bar{\partial}^{n-1} k_0 \in S_\theta \overline{\text{Ran } A} \subset \overline{\text{Ran } A} + \langle k_0, g \rangle = \overline{\text{Ran } A} + \langle g \rangle$$

for some $g \in K_\theta$ such that $S_\theta \overline{\text{Ran } A} \subset \overline{\text{Ran } A} + \langle k_0, g \rangle$. By the construction we have $\bar{\partial}^n k_0 \notin \overline{\text{Ran } A}$. Comparing this with (27), we get $g \in \overline{\text{Ran } A} + \langle \bar{\partial}^n k_0 \rangle$ and thus

$$S_\theta \overline{\text{Ran } A} \subset \overline{\text{Ran } A} + \langle k_0, g \rangle \subset \overline{\text{Ran } A} + \langle \bar{\partial}^n k_0 \rangle.$$

Hence the Proposition is proved in the case $\theta(0) = 0$. The general situation can be reduced to this case by using Propositions 2.3 and 2.4. \square

Remark. Truncated Toeplitz operators are symmetric with respect to the conjugation C , see Section 2.1. Using this fact, one can obtain an analogue of Proposition 3.1 for $\overline{\text{Ran } A}$ with S_θ^* and $\partial^m \tilde{k}_0$ in place of S_θ and $\bar{\partial}^n k_0$.

Let $U_\alpha = S_\theta + c_\alpha k_0 \otimes \tilde{k}_0$ be the Clark unitary perturbation of S_θ . Consider the embedding $V_\alpha : K_\theta \rightarrow L^2(\sigma_\alpha)$ from Section 2.5.

Proposition 3.4. *Let $A \in \mathcal{T}_\theta$, and assume that $\overline{\text{Ran } A} \neq K_\theta$. Set $F = V_\alpha \overline{\text{Ran } A}$. Then $zF \subset F + \langle z^n \rangle$, where $n \in \mathbb{Z}_+$ is the maximal integer such that $z^j \in F$ for every $0 \leq j < n$.*

Proof. By Proposition 3.1 we have $S_\theta \overline{\text{Ran } A} \subset \overline{\text{Ran } A} + \langle \bar{\partial}^n k_0 \rangle$, where $n \in \mathbb{Z}_+$ is the maximal integer such that $\partial^j k_0 \in \overline{\text{Ran } A}$ for every $0 \leq j < n$. Hence $U_\alpha \overline{\text{Ran } A} \subset \overline{\text{Ran } A} + \langle \bar{\partial}^n k_0 \rangle$. It remains to apply the operator V_α to both sides of this inclusion and use Proposition 2.7. \square

Problem. *Given a finite Borel measure ν supported on the unit circle \mathbb{T} , to describe all subspaces $F \subset L^2(\mathbb{T}, \nu)$ such that $zF \subset F \dot{+} \langle 1 \rangle$.*

In the next section we treat the finite-dimensional case of this problem.

4. PROOF OF THEOREM 1

Hereinafter we assume that the space K_θ has infinite dimension (equivalently, the inner function θ is not a finite Blaschke product). The case $\dim K_\theta < \infty$ is considered in [1], where the following fact is proved: every truncated Toeplitz operator on K_θ , $\dim K_\theta < \infty$, is a finite linear combination of the rank-one operators (3).

Theorem 1 follows from Lemmas 1.1, 1.2, 1.3. We now turn to proving this results.

4.1. Proof of Lemma 1.1. The proof is based on the following proposition.

Proposition 4.1. *Let σ be a finite Borel measure supported on the unit circle \mathbb{T} . Suppose that $F \subset L^2(\sigma)$ is a finite-dimensional subspace satisfying $zF \subset F \dot{+} \langle 1 \rangle$. If F does not contain indicators of singletons, then there exists a finite collection of points $\lambda_k \in \mathbb{C}$ such that $F = Q(\lambda_1, p_1) \dot{+} Q(\lambda_2, p_2) \dot{+} \dots \dot{+} Q(\lambda_s, p_s)$, where $Q(\lambda_k, p_k) = \text{span}\{(z - \lambda_k)^{-j}, j = 1 \dots p_k\}$, $k = 1 \dots s$.*

Proof. Denote by \mathcal{P} the non-orthogonal projection on $F \dot{+} \langle 1 \rangle$ with the range F and the kernel $\langle 1 \rangle$. Let M_z be the operator of multiplication by the independent variable on $L^2(\sigma)$. The finite-rank operator $T = \mathcal{P}M_z : F \rightarrow F$ has complete family of root vectors. Consider the root subspace $G_\lambda = \text{Ker}(T - \lambda I)^p$, $G_\lambda \neq \text{Ker}(T - \lambda I)^{p-1}$. The Proposition will be proved as soon as we show that $G_\lambda = Q(\lambda, p)$.

Case 1. $\lambda \in \mathbb{T}$, $\sigma(\{\lambda\}) > 0$.

Take a vector $f \in \text{Ker}(T - \lambda I) \subset G_\lambda$. We have $(z - \lambda)f = c$ for a constant $c \in \mathbb{C}$. Therefore, f is a scalar multiple of the indicator of the singleton $\{\lambda\}$. But that contradicts the hypothesis. Hence this case does not arise.

Case 2. $\lambda \in \mathbb{C}$, $\sigma(\{\lambda\}) = 0$.

Take a vector $f \in G_\lambda$ such that $f_1 = (T - \lambda I)^{p-1}f \neq 0$. We have $(T - \lambda I)f_1 = 0$ by the construction. On the other hand, we have $(T - \lambda I)f_1 = (z - \lambda)f_1 - c_1$ for some constant $c_1 \in \mathbb{C}$. Taking into account $\lambda(\{\lambda\}) = 0$, we see that

$$f_1 = \frac{c_1}{z - \lambda}.$$

If $p = 1$, we stop the procedure. Otherwise put $f_2 = (T - \lambda I)^{p-2}f$ and consider $(T - \lambda I)f_2 = (z - \lambda)f_2 - c_2$. Since $(T - \lambda I)f_2 = f_1$, we have

$$f_2 = \frac{c_1}{(z - \lambda)^2} + \frac{c_2}{z - \lambda}.$$

Continuing this procedure, we obtain

$$f = f_p = \frac{c_1}{(z - \lambda)^p} + \frac{c_2}{(z - \lambda)^{p-1}} + \dots + \frac{c_p}{z - \lambda}.$$

Since $c_1 \neq 0$ and $f_j \in G_\lambda$ for every $j = 1 \dots p$, we get $(z - \lambda)^{-j} \in G_\lambda$ and hence $Q(\lambda, p) \subset G_\lambda$. Now take an arbitrary vector $f \in G_\lambda$ and find a number r , $1 \leq r \leq p$, such that $f \in \text{Ker}(T - \lambda I)^r$ but $f_1 = (T - \lambda I)^{r-1}f \neq 0$. Arguing as above, we see that $f \in Q(\lambda, r) \subset Q(\lambda, p)$ and thus $G_\lambda \subset Q(\lambda, p)$. \square

Proof (Lemma 1.1). Let A be a finite-rank truncated Toeplitz operator on K_θ . Since the subspace $\text{Ran } A$ has finite dimension, it cannot contain an infinite system of reproducing kernels, see Proposition 2.8. Therefore, by Proposition 2.5 we can choose the Clark measure σ_α so that the subspace $F = V_\alpha \text{Ran } A$ of the space $L^2(\sigma_\alpha)$ does not contain indicators of singletons. It follows from Proposition 3.4 that $zF \subset F \dot{+} \langle z^n \rangle$, where $n \in \mathbb{Z}_+$ is the maximal integer such that $z^j \in F$ for every $0 \leq j < n$. The subspace $\bar{z}^n F$ satisfies the assumptions of Proposition 4.1. We have

$$F = z^n(Q(\lambda_1, p_1) \dot{+} Q(\lambda_2, p_2) \dot{+} \dots \dot{+} Q(\lambda_s, p_s)).$$

In the case $n = 0$, the application of Propositions 2.6 and 2.7 concludes the proof. Now assume that $n \geq 1$. Since $z^j \in F$ for every integer $0 \leq j < n$, we necessarily have $\lambda_{k_0} = 0$, $p_{k_0} = n$ for some $1 \leq k_0 \leq s$. Renumber the sequence $\{\lambda_k\}$ so that $\lambda_1 = 0$. The subspace $z^n Q(0, n)$ is the set of polynomials of degree at most $n - 1$. A simple algebra gives us

$$F = z^n Q(0, n) \dot{+} Q(\lambda_2, p_2) \dot{+} \dots \dot{+} Q(\lambda_s, p_s).$$

Now the result follows from Propositions 2.6 and 2.7. \square

4.2. Proof of Lemma 1.2. Before the proof we need the following general result.

Proposition 4.2. *Let $n \in \mathbb{Z}_+$, $A \in \mathcal{T}_\theta$. Suppose that $\overline{\text{Ran } A} = F_1 \dot{+} F_2$, where F_1 and F_2 are subspaces of K_θ such that*

$$(28) \quad S_\theta F_1 \subset F_1 \dot{+} \langle k_0 \rangle; \quad S_\theta F_2 \subset F_2 \dot{+} \langle \bar{\partial}^n k_0 \rangle, \quad \bar{\partial}^j k_0 \in F, \quad 0 \leq j < n.$$

Let also $\bar{\partial}^n k_0 \notin \overline{\text{Ran } A}$. Then $A = A_1 + A_2$, where $A_k \in \mathcal{T}_\theta$ and $\overline{\text{Ran } A_k} = F_k$, $k = 1, 2$.

Proof. Denote by \mathcal{P} the non-orthogonal projection on $F_1 \dot{+} F_2 \dot{+} \langle \bar{\partial}^n k_0 \rangle$ with the range $F_2 \dot{+} \langle \bar{\partial}^n k_0 \rangle$ and the kernel F_1 . We want to show that $A_2 = \mathcal{P}A \in \mathcal{T}_\theta$. By Theorem 3, we need to check that

$$(29) \quad A_2 - S_\theta A_2 S_\theta^* = \psi \otimes k_0 + k_0 \otimes \chi$$

for some $\psi, \chi \in K_\theta$. We have

$$A_2 - S_\theta A_2 S_\theta^* = \mathcal{P}(A - S_\theta A S_\theta^*) + (\mathcal{P}S_\theta - S_\theta \mathcal{P})A S_\theta^*.$$

Since A is a truncated Toeplitz operator, it satisfies (24) with $\psi_1, \chi_1 \in K_\theta$. Hence,

$$\mathcal{P}(A - S_\theta A S_\theta^*) = (\mathcal{P}\psi_1) \otimes k_0 + (\mathcal{P}k_0) \otimes \chi_1 = (\mathcal{P}\psi_1) \otimes k_0 + k_0 \otimes \chi_1.$$

Next, the operator $\mathcal{P}S_\theta - S_\theta \mathcal{P}$ vanishes on F_2 and maps F_1 to $\langle k_0 \rangle$. Therefore, we have $\text{Ran}(\mathcal{P}S_\theta - S_\theta \mathcal{P})A S_\theta^* \subset \langle k_0 \rangle$. This proves (29). Now put $A_1 = A - A_2$ and obtain the required representation. \square

Proof (Lemma 1.2). It follows from Proposition 2.2 that any splitting of the sum in (5) into two summands gives us subspaces F_1, F_2 with property (28). Consequently applying Proposition 4.2, we obtain the required.

4.3. Proof of Lemma 1.3. Let A be a truncated Toeplitz operator on K_θ with the range $\text{Ran } A = F(\lambda, n)$, where $\lambda \in \mathbb{D} \cup \Omega_n \cup \mathbb{D}_e$. Truncated Toeplitz operators are complex symmetric with respect to the conjugation C , see Section 2.1. Hence

$$\begin{aligned} \text{Ran } A &= \text{span}\{\partial^j \tilde{k}_\lambda^*, j = 0 \dots n\} \quad \text{in the case } \lambda \in \mathbb{D}_e, \\ \text{Ran } A^* &= \text{span}\{\partial^j \tilde{k}_\lambda, j = 0 \dots n\} \quad \text{in the case } \lambda \in \mathbb{D} \cup \Omega_n. \end{aligned}$$

Passing if necessary to the adjoint operator, we can assume that

$$\text{Ran } A = \text{span}\{\partial^j \tilde{k}_\mu, j = 0 \dots n\}$$

for some point $\mu \in \mathbb{D} \cup \Omega_n$. Then $\text{Ran } A^* = \text{span}\{\bar{\partial}^j k_\mu, j = 0 \dots n\}$. Every such operator has the form

$$(30) \quad A = \sum_{0 \leq p, q \leq n} a_{p,q} (\partial^p \tilde{k}_\mu \otimes \bar{\partial}^q k_\mu)$$

for some coefficients $a_{p,q} \in \mathbb{C}$. We claim that $A = \sum_{s=0}^n a_{0,s} \cdot D^s [\tilde{k}_\mu \otimes k_\mu]$. Consider firstly the case $\mu \neq 0$. Set $T_{pq} = \partial^p \tilde{k}_\mu \otimes \bar{\partial}^q k_\mu$. For $1 \leq p, q \leq n$ we have

$$\begin{aligned} (31) \quad T_{pq} - S_\theta T_{pq} S_\theta^* &= \partial^p \tilde{k}_\mu \otimes \bar{\partial}^q k_\mu - \partial^p (S_\theta \tilde{k}_\mu) \otimes \bar{\partial}^q (S_\theta k_\mu) \\ &= \partial^p \tilde{k}_\mu \otimes \bar{\partial}^q k_\mu - \partial^p (\mu \tilde{k}_\mu - \theta(\mu) k_0) \otimes \bar{\partial}^q ((k_\mu - k_0)/\bar{\mu}) \\ &= \partial^p \tilde{k}_\mu \otimes \bar{\partial}^q k_\mu - \partial^p (\mu \tilde{k}_\mu) \otimes \bar{\partial}^q (k_\mu/\bar{\mu}) + Z_{pq}, \end{aligned}$$

where Z_{pq} is an operator of the form $\psi_{pq} \otimes k_0 + k_0 \otimes \chi_{pq}$. Using the identity $(zf)^{(p)} = pf^{(p-1)} + zf^{(p)}$, we get

$$\partial^p (\mu \tilde{k}_\mu) = p\partial^{p-1} \tilde{k}_\mu + \mu \partial^p \tilde{k}_\mu, \quad \bar{\partial}^q k_\mu = \bar{\partial}^q (\bar{\mu}(k_\mu/\bar{\mu})) = q\bar{\partial}^{q-1} (k_\mu/\bar{\mu}) + \bar{\mu} \bar{\partial}^q (k_\mu/\bar{\mu}).$$

Substituting this into (31), we obtain

$$(32) \quad T_{pq} - S_\theta T_{pq} S_\theta^* = q \left(\partial^p \tilde{k}_\mu \otimes \bar{\partial}^{q-1} (k_\mu/\bar{\mu}) \right) - p \left(\partial^{p-1} \tilde{k}_\mu \otimes \bar{\partial}^q (k_\mu/\bar{\mu}) \right) + Z_{pq}.$$

For the operators T_{00} , T_{01} and T_{10} we have

$$\begin{aligned} (33) \quad T_{00} - S_\theta T_{00} S_\theta^* &= Z_{00}; \\ T_{01} - S_\theta T_{01} S_\theta^* &= \tilde{k}_\mu \otimes (k_\mu/\bar{\mu}) + Z_{01}; \\ T_{10} - S_\theta T_{10} S_\theta^* &= -\tilde{k}_\mu \otimes (k_\mu/\bar{\mu}) + Z_{10}. \end{aligned}$$

Since A is a truncated Toeplitz operator, it satisfies (24) with some $\psi, \chi \in K_\theta$. Combining (24), (32) and (33), we obtain

$$(34) \quad \Psi \otimes k_0 + k_0 \otimes \Phi = \sum_{0 \leq p, q \leq n} ((q+1)a_{p,q+1} - (p+1)a_{p+1,q}) \cdot \left(\partial^p \tilde{k}_\mu \otimes \bar{\partial}^q (k_\mu/\bar{\mu}) \right),$$

where $\Psi, \Phi \in K_\theta$ and $a_{n+1,q} = a_{p,n+1} = 0$ for all $0 \leq p, q \leq n$. It follows from Proposition 2.8 that

$$(35) \quad (q+1)a_{p,q+1} - (p+1)a_{p+1,q} = 0, \quad 0 \leq p, q \leq n.$$

In the above formula there is no restriction on $a_{0,0}$, which agrees well with the fact that $\tilde{k}_\mu \otimes k_\mu \in \mathcal{T}_\theta$. For each $1 \leq s \leq n$, from (35) we get the following system:

$$(36) \quad \left. \begin{aligned} sa_{0,s} - a_{1,s-1} &= 0 \\ (s-1)a_{1,s-1} - 2a_{2,s-2} &= 0 \\ &\dots \\ 2a_{s-2,1} - (s-1)a_{s-1,1} &= 0 \\ a_{s-1,1} - sa_{s,0} &= 0 \end{aligned} \right\}$$

Solving this system, we obtain

$$a_{t,s-t} = a_{0,s} C_{s-t}^t, \quad C_{s-t}^t = \frac{s!}{t!(s-t)!}, \quad 0 \leq t \leq s.$$

It follows from (35) that $na_{n,n} - (n+1)a_{n+1,n-1} = 0$ and thus $a_{n,n} = 0$. By induction, we have $a_{p,q} = 0$ for all indexes p, q such that $p+q > n$. Now we get the required representation from formulas (11) and (30):

$$(37) \quad A = \sum_{s=0}^n \sum_{t=0}^s a_{t,s-t} \left(\partial^t \tilde{k}_\mu \otimes \bar{\partial}^{s-t} k_\mu \right) = \sum_{s=0}^n a_{0,s} \cdot D^s [\tilde{k}_\mu \otimes k_\mu].$$

In the case $\mu = 0$, put $T_{pq} = \partial^p \tilde{k}_0 \otimes \bar{\partial}^q k_0$. It follows from Proposition 2.1 that

$$\begin{aligned} T_{pq} - S_\theta T_{pq} S_\theta^* &= \partial^p \tilde{k}_0 \otimes \bar{\partial}^q k_0 - \frac{p}{q+1} \partial^{p-1} \tilde{k}_0 \otimes \bar{\partial}^{q+1} k_0 + Z_{pq}, \quad p \geq 1; \\ T_{pq} - S_\theta T_{pq} S_\theta^* &= \partial^p \tilde{k}_0 \otimes \bar{\partial}^q k_0 + Z_{pq}, \quad p = 0. \end{aligned}$$

Proceeding as in the case $\mu \neq 0$, we obtain the system

$$a_{p,q} - \frac{p+1}{q} a_{p+1,q-1} = 0, \quad 0 \leq p \leq n, \quad 1 \leq q \leq n+1,$$

where $a_{n+1,q} = 0$ for all q and $a_{p,-1} = 0$ for all p . This system has the same solution as the system in (35). Hence we have representation (37) in the case $\mu = 0$ as well. \square

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